Non-Linear Effects in a Yamabe-Type Problem with Quasi-Linear Weight

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Abstract

We study the quasi-linear minimization problem on $H_0^1(\Omega) \subset L^q$ with $q = \frac{2n}{n-2}$:

$$\inf_{\|u\|_{L^q}=1} \int_\Omega (1 + |x|^\beta |u|^k)|\nabla u|^2.$$

We show that minimizers exist only in the range $\beta < kn/q$ which corresponds to a dominant non-linear term. On the contrary, the linear influence for $\beta \geq kn/q$ prevents their existence.

1 Introduction

Given a smooth bounded open subset $\Omega \subset \mathbb{R}^n$ with $n \geq 3$, let us consider the minimizing problem

$$S_\Omega(\beta, k) \quad = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^q(\Omega)}=1}} I_{\Omega;\beta,k}(u) \qquad \text{with} \qquad I_{\Omega;\beta,k}(u) = \int_\Omega p(x, u(x))|\nabla u(x)|^2 \, dx \tag{1}$$

and $p(x, y) = 1 + |x|^\beta |y|^k$. Here $q = \frac{2n}{n-2}$ denotes the critical exponent of the Sobolev injection $H_0^1(\Omega) \subset L^q(\Omega)$. We restrict ourselves to the case $\beta \geq 0$ and $0 \leq k \leq q$. The Sobolev injection for $u \in H^{s+1}(\Omega)$ and $\nabla u \in H^s(\Omega)$ gives :

$$I_{\Omega;\beta,k}(u) \leq \|u\|^2_{H_0^1(\Omega)} + C_s \left(\sup_{x\in\Omega} |x|^\beta \right) \|u\|^2_{H^{s+1}(\Omega)} \quad \text{for} \quad s \geq \frac{kn}{q(k+2)}$$

so $I_{\Omega;\beta,k}(u) < \infty$ on a dense subset of $H_0^1(\Omega)$. Note in particular that one can have $I_{\Omega;\beta,k}(u) < \infty$ without having $u \in L^\infty_{\text{loc}}(\Omega)$. If $0 \notin \bar{\Omega}$, the problem is essentially equivalent to the case $\beta = 0$ thus one will also assume from now on that $0 \in \Omega$. The case $0 \in \partial\Omega$ is interesting but will not be addressed in this paper.

As $|\nabla|u|| = |\nabla u|$ for any $u \in H_0^1(\Omega)$, one has

$$I_{\Omega;\beta,k}(u) = I_{\Omega;\beta,k}(|u|) \tag{2}$$

thus, when dealing with (1), one can assume without loss of generality that $u \geq 0$.

The Euler-Lagrange equation formally associated to (1) is

$$\begin{cases} -\operatorname{div}\big(p(x,u(x))\nabla u\big) + Q(x,u(x))|\nabla u(x)|^2 = \mu|u(x)|^{q-2}u(x) & \text{in } \Omega \\ u \geq 0 & \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

with $Q(x,y) = \frac{k}{2}|x|^\beta|y|^{k-2}y$ and $\mu = S_\Omega(\beta,k)$. However, the logical relation between (1) and (3) is subtle : $I_{\Omega;\beta,k}$ is not Gateaux differentiable on $H_0^1(\Omega)$ because one can only expect $I_{\Omega;\beta,k}(u) = +\infty$ for a general function $u \in H_0^1(\Omega)$. However, if a minimizer u of (1) belongs to $H_0^1 \cap L^\infty(\Omega)$ then, without restriction, one can assume $u \geq 0$ and for any test-function $\phi \in H_0^1 \cap L^\infty(\Omega)$, one has

$$\forall t \in \mathbb{R}, \qquad I_{\Omega;\beta,k}\left(\frac{u+t\phi}{\|u+t\phi\|_{L^q}}\right) < \infty.$$

A finite expansion around $t=0$ then gives (3) in the weak sense, with the test-function ϕ.

The following generalization of (1) will be adressed in a subsequent paper :

$$S_\Omega(\lambda;\beta,k) \quad = \inf_{\substack{u\in H_0^1(\Omega)\\ \|u\|_{L^q(\Omega)}=1}} J_{\Omega;\beta,k}(\lambda,u) \qquad \text{with} \qquad J_{\Omega;\beta,k}(\lambda,u) = I_{\Omega;\beta,k}(u) - \lambda\int_\Omega |u|^2. \tag{4}$$

for $\lambda > 0$, which is a compact perturbation of the case $\lambda = 0$.

This type of problem is inspired by the study of the Yamabe problem which has been the source of a large literature. The Yamabe invariant of a compact Riemannian manifold (M,g) is :

$$\mathcal{Y}(M) = \inf_{\substack{\phi\in C^\infty(M;\mathbb{R}_+)\\ \|\phi\|_{L^q(M)}=1}} \int_M \left(4\tfrac{n-1}{n-2}|\nabla\phi|^2 + \sigma\phi^2\right) dV_g$$

where ∇ denotes the covariant derivative with respect to g and σ is the scalar curvature of g ; $\mathcal{Y}(M)$ is an invariant of the conformal class $\mathcal{C}$ of (M,g). One can check easily that $\mathcal{Y}(M) \leq \mathcal{Y}(\mathbb{S}^n)$. The so called *Yamabe problem* which is the question of finding a manifold in $\mathcal{C}$ with constant scalar curvature can be solved if $\mathcal{Y}(M) < \mathcal{Y}(\mathbb{S}^n)$. In dimension $n \geq 6$, one can show that unless M is conformal to the standard sphere, the strict inequality holds using a "local" test function ϕ ; however, for $n \leq 5$, one must use a "global" test function (see [11] for an in-depth review of this historical problem and more precise statements).

Even though problems (1) and (4) seem of much less geometric nature, they should be considered as a toy model of the Yamabe problem that can be played with in $\mathbb{R}^n$. As it will be shown in this paper, those toy models retain some interesting properties from their geometrical counterpart : the functions u_ε that realise the infimum $\mathcal{Y}(\mathbb{S}^n)$ still play a crucial role in (1) and (4) and the existence of a solution is an exclusively non-linear effect.

Another motivation can be found in the line of [4] for the study of sharp Sobolev and Gagliardo-Nirenberg inequalities. For example, among other striking results it is shown that, for an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^n$:

$$\inf_{\|u\|_{L^q}=1} \int_{\mathbb{R}^n} \|\nabla u(x)\|^2\,dx = \|\nabla h\|_{L^2} \qquad \text{with} \qquad h(x) = \frac{1}{(c+\|x\|^2)^{\frac{n-2}{2}}}$$

and a constant c such that $\|h\|_{L^q} = 1$. The problem (1) can be seen as a quasi-linear generalisation where the norm $\|\cdot\|$ measuring ∇u is allowed to depend on u itself.

1.1 Bibliographical notes

The case $\beta = k = 0$ *i.e.* a constant weight $p(x,y) = 1$ has been addressed in the celebrated [2] where it is shown in particular that the equation

$$-\Delta u = u^{q-1} + \lambda u, \qquad u > 0 \tag{5}$$

has a solution $u \in H_0^1(\Omega)$ if $n \geq 4$ and $0 < \lambda < \lambda_1(\Omega) = \inf\limits_{u \in H_0^1(\Omega)\setminus\{0\}} \dfrac{I_{\Omega;0,0}(u)}{\int_\Omega |u|^2 dx}$.

On the contrary, for $\lambda = 0$, the problem (5) has no solution if Ω is star-shaped around the origin. In dimension $n = 3$, the situation is more subtle. For example, if $\Omega = \{x \in \mathbb{R}^3 \,;\, |x| < 1\}$, then (5) admits solutions for $\lambda \in]\frac{\pi^2}{4}, \pi^2[$ but has none if $\lambda \in]0, \frac{\pi^2}{4}[$. See also [6] for the behavior of solutions when $\lambda \to (\pi^2/4)_+$ and for generalizations to general domains.

A first attempt to the case $\beta \neq 0$ but with $k = 0$ (*i.e.* a weight that does not depend on u, which is the semi-linear case) was achieved in [10]. More precisely, [10] deals with a weight $p \in H^1(\Omega) \cap C(\bar{\Omega})$ that admits a global minimum of the form

$$p(x) = p_0 + c|x-a|^\beta + o\left(|x-a|^\beta\right), \qquad c > 0.$$

They show that for $n \geq 3$ and $\beta > 0$, there exists $\lambda_0 \geq 0$ such that (4) admits a solution for any $\lambda \in]\lambda_0, \lambda_1[$ where λ_1 is the first eigenvalue of the operator $-\operatorname{div}(p(x)\nabla\cdot)$ in Ω, with Dirichlet boundary conditions (and for $n \geq 4$ and $\beta > 2$, one can check that $\lambda_0 = 0$). On the contrary, the problem (4) admits no solution if $\lambda \leq \lambda_0'$ for some $\lambda_0' \in [0; \lambda_0]$ or for $\lambda \geq \lambda_1$. See [10] for more precise statements.

Similarly, the semi-linear case in which the minimum value of the weight is achieved in more than one point was studied in [9] ; namely in dimension $n \geq 4$ if

$$p^{-1}\left(\inf_{x\in\Omega} p(x)\right) = \{a_0, a_1, \ldots, a_N\}$$

then multiple solutions that concentrate around each of the a_j can be found for $\lambda > 0$ small enough.

For $\lambda = 0$ and a star-shaped domain, it is well known (see [2]) that the linear problem $\beta = k = 0$ has no solution. However, when the topology of the domain is not trivial, the problem (1) has at least one solution (see [5] for $\beta = k = 0$; [8] and [10] for $k = 0$, $\beta \neq 0$).

1.2 Ideas and main results

In this article, the introduction of the fully quasi-linear term $|x|^\beta |u|^k$ in (1) provides a more unified approach and generates a sharp contrast between sub- and super-critical cases. Moreover, the existence of minimizers will be shown to occur exactly in the sub-cases where the nonlinearity is dominant.

The critical exponent for (1) can be found by the following scaling argument. As $0 \in \Omega$, the non-linear term tends to concentrate minimizing sequences around $x = 0$. Let us therefore consider the blow-up of $u \in H_0^1(\Omega)$ around $x = 0$. This means one looks at the function v_ε defined by :

$$\forall \varepsilon > 0, \qquad u(x) = \varepsilon^{-n/q} v_\varepsilon(x/\varepsilon). \tag{6}$$

One has $v_\varepsilon \in H_0^1(\Omega_\varepsilon)$ with $\Omega_\varepsilon = \{\varepsilon^{-1} y \,;\, y \in \Omega\}$ and $\|v_\varepsilon\|_{L^q(\Omega_\varepsilon)} = \|u\|_{L^q(\Omega)}$. Moreover, the definition of q ensures that $2 - n + \frac{2n}{q} = 0$, thus :

$$I_{\Omega;\beta,k}(u) = \int_{\Omega_\varepsilon} \left(1 + \varepsilon^{\beta - \frac{kn}{q}} |y|^\beta |v_\varepsilon(y)|^k\right) |\nabla v_\varepsilon(y)|^2 \, dy. \tag{7}$$

Depending on the ratio β/k, different situations occur.

- If $\frac{\beta}{k} < \frac{n}{q}$ leading term of the blow-up around $x = 0$ is

$$I_{\Omega;\beta,k}(u) \underset{\varepsilon\to 0}{\sim} \varepsilon^{-\left(\frac{kn}{q}-\beta\right)} \int_{\Omega_\varepsilon} |y|^\beta |v_\varepsilon(y)|^k |\nabla v_\varepsilon(y)|^2 dy.$$

  One can expect the effect of the non-linearity to be dominant and one will show in this paper that (1) admits indeed minimizers in this case.

- If $\frac{\beta}{k} = \frac{n}{q}$ both terms have the same weight and

$$\forall \varepsilon > 0, \qquad I_{\Omega;\beta,k}(u) = I_{\Omega_\varepsilon;\beta,k}(v_\varepsilon).$$

  One will show that, similarly to the classical case $\beta = k = 0$, the corresponding infimum $S(\beta, k)$ does not depends on Ω and that (1) admits no smooth minimizer.

- If $\frac{\beta}{k} > \frac{n}{q}$, the blow-up around 0 gives

$$I_{\Omega;\beta,k}(u) \underset{\varepsilon\to 0}{\sim} \int_{\Omega_\varepsilon} |\nabla v_\varepsilon(y)|^2 dy.$$

  In this case, one can show that the linear behavior is dominant and that (1) admits no minimizer. Moreover, one can find a common minimizing sequences for both the linear and the non-linear problem. A cheap way to justify this is as follows. The problem (1) tends to concentrate u as a radial decreasing function around the origin. Thus, when $\beta/k > n/q$, one can expect $|u(x)|^q \ll 1/|x|^{\beta q/k}$ because the right-hand side would not be locally integrable while the left-hand side is required to. In turn, this inequality reads $|x|^\beta |u(x)|^k \ll 1$ which eliminates the non-linear contribution in the minimizing problem (1).

The infimum for the classical problem with $\beta = k = 0$ is (see *e.g.* [2]) :

$$S = \inf_{\substack{w\in H_0^1(\Omega)\\ \|w\|_{L^q}=1}} \int_\Omega |\nabla w|^2 \tag{8}$$

which does not depend on Ω. Let us now state the main Theorem concerning (1).

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ a smooth bounded domain with $n \geq 3$ and $q = \frac{2n}{n-2}$ the critical exponent for the Sobolev injection $H_0^1(\Omega) \subset L^q(\Omega)$.*

1. *If $0 \leq \beta < kn/q$ then $S_\Omega(\beta, k) > S$ and the infimum for $S_\Omega(\beta, k)$ is achieved.*
2. *If $\beta = kn/q$ then $S_\Omega(\beta, k)$ does not depend on Ω and $S_\Omega(\beta, k) \geq S$. Moreover, if Ω is star-shaped around $x = 0$, then the minimizing problem (1) admits no minimizers in the class :*

$$H_0^1 \cap H^{3/2} \cap L^\infty(\Omega).$$

   If $k < 1$, the negative result holds, provided additionally $u^{k-1} \in L^n(\Omega)$.
3. *If $\beta > kn/q$ then $S_\Omega(\beta, k) = S$ and the infimum for $S_\Omega(\beta, k)$ is not achieved in $H_0^1(\Omega)$.*

Remarks.

1. In the first case, one has $k > 0$, thus results concerning $k = 0$ (such as those of *e.g.* [9] and [10]) are included either in our second or third case.
2. If the minimizing problem (1) is achieved for $u \in H_0^1(\Omega)$, then $|u|$ is a positive minimizer. In particular, if $\beta < kn/q$, the problem always admits positive minimizers.
3. In the critical case, it is not known wether a non-smooth minimizer could exist in $H_0^1 \backslash (H^{3/2} \cap L^\infty)$. Such a minimizer could have a non-constant sign.

1.3 Structure of the article

Each of the following sections deals with one sub-case $\beta \lesseqgtr kn/q$.

2 Subcritical case $(0 \le \beta < kn/q)$: existence of minimizers

The case $\beta < kn/q$ is especially interesting because it reveals that the non-linear weight $|u|^k$ helps for the existence of a minimizer. Note that $k > 0$ in throughout this section.

Proposition 2 *If $0 \le \frac{\beta}{k} < \frac{n}{q}$, the minimization problem* (1) *has at least one solution $u \in H_0^1(\Omega)$. Moreover, one has*

$$S_\Omega(\beta, k) > S \tag{9}$$

where S is defined by (8).

Proof. Let us prove first that the existence of a solution implies the strict inequality in (9). By contradiction, if $S_\Omega(\beta, k) = S$ and if u is a minimizer for (1) thus $u \not\equiv 0$, one has

$$S = \int_\Omega (1 + |x|^\beta |u(x)|^k)|\nabla u(x)|^2 dx > \int_\Omega |\nabla u(x)|^2 dx$$

which contradicts the definition of S. Thus, if the minimization problem has a solution, the strict inequality (9) must hold.

Let us prove now that (1) has at least one solution $u \in H_0^1(\Omega)$. Let $(u_j)_{j\in\mathbb{N}} \in H_0^1(\Omega)$ be a minimizing sequence for (1), *i.e.* :

$$I_{\Omega;\beta,k}(u_j) = S_\Omega(\beta, k) + o(1), \qquad \text{and} \qquad \|u_j\|_{L^q} = 1.$$

As noticed in the introduction, one can assume without restriction that $u_j \ge 0$. Up to a subsequence, still denoted by u_j, there exists $u \in H_0^1(\Omega)$ such that $u_j(x) \to u(x)$ for almost every $x \in \Omega$ and such that :

$$\begin{aligned} u_j \rightharpoonup u \quad &\text{weakly in} \quad H_0^1 \cap L^q(\Omega),\\ u_j \to u \quad &\text{strongly in} \quad L^\ell(\Omega) \text{ for any } \ell < q. \end{aligned}$$

The idea of the proof is to introduce $v_j = u_j^{\frac{k}{2}+1}$ and prove that v_j is a bounded sequence in $W_0^{1,r} \subset L^p$ for indices r and p such that

$$p\left(\frac{k}{2} + 1\right) \ge q.$$

The key point is the formula :

$$I_{\Omega;\beta,k}(u_j) = \int_\Omega |\nabla u_j|^2 + \left(\frac{k}{2} + 1\right)^{-2} \int_\Omega |x|^\beta |\nabla v_j|^2 \tag{10}$$

which gives "almost" an H_0^1 bound on v_j (and does indeed if $\beta = 0$). For $r \in [1, 2[$, one has :

$$\int_\Omega |\nabla v_j|^r \le \left(\int_\Omega |x|^\beta |\nabla v_j|^2 dx\right)^{r/2} \left(\int_\Omega |x|^{-\frac{\beta r}{r-2}} dx\right)^{1-r/2}$$

The integral in the right-hand side is bounded provided $\frac{\beta r}{r-2} < n$. All the previous conditions are satisfied if one can find r such that :

$$1 \le r < 2, \qquad \beta < n\left(\frac{2}{r} - 1\right), \qquad \frac{k}{2} + 1 \ge \frac{q}{p} = q\left(\frac{1}{r} - \frac{1}{n}\right).$$

This system of inequalities boils down to :

$$1 \le r < 2, \qquad \frac{\beta}{n} < \frac{2}{r} - 1 \le \frac{2}{q}\left(\frac{k}{2} + 1 + \frac{q}{n}\right) - 1$$

which is finally equivalent to $\beta < kn/q$ provided $k \le q$. Using the compacity of the inclusion $W_0^{1,r} \subset L^p$ and up to a subsequence, one has $v_j \to v = u^{\frac{k}{2}+1}$ strongly in L^p. Finally, as $u_j \ge 0$ and $u \ge 0$, one has :

$$|u_j - u|^q \le C\left|u_j^q - u^q\right| = C\left|v_j^{q/(k/2+1)} - v^{q/(k/2+1)}\right|$$

and thus $u_j \to u$ strongly in L^q. One gets $\|u\|_{L^q} = 1$. The following compacity result then implies that u is a minimizer for (1). ■

Proposition 3 *If $u_j \in H_0^1(\Omega)$ is a minimizing sequence for* (1) *with $\|u_j\|_{L^q(\Omega)} = 1$ and such that*

$$u_j \to u \qquad \text{in } L^2(\Omega), \qquad \text{and} \qquad \nabla u_j \rightharpoonup \nabla u \quad \text{weakly in } L^2(\Omega),$$

the weak limit $u \in H_0^1(\Omega)$ is a minimizer of the problem (1) *if and only if $\|u\|_{L^q(\Omega)} = 1$.*

Proof. It is an consequence of the main Theorem of [7, p. 77] (see also [14]) applied to the function :

$$f(x, z, p) = (1 + |x|^\beta |z|^k)|p|^2$$

which is positive, measurable on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, continuous with respect to z, convex with respect to p. Then

$$I(u) = \int_\Omega f(x, u, \nabla u) \le \liminf_{j\to\infty} \int_\Omega f(x, u_j, \nabla u_j) = \liminf_{j\to\infty} I(u_j).$$

If u_j is a minimizing sequence, then $I(u) = S_\Omega(\beta, k)$ and u is a minimizer if and only if $\|u\|_{L^q} = 1$. ■

Remarks

- The sequence u_j converges strongly in $H_0^1(\Omega)$ towards u because $\nabla u_j \rightharpoonup \nabla u$ weakly in $L^2(\Omega)$ and :
$$\int_\Omega |\nabla u_j|^2 - \int_\Omega |\nabla u|^2 = I(u_j) - I(u) + \int_\Omega |x|^\beta u^k |\nabla u|^2 - \int_\Omega |x|^\beta u_j^k |\nabla u_j|^2.$$
Applying the previous lemma with $\tilde{f}(x, z, p) = |x|^\beta |z|^k |p|^2$ provides
$$\forall j \in \mathbb{N}, \qquad \int_\Omega |\nabla u_j|^2 \le \int_\Omega |\nabla u|^2 + o(1)$$
and Fatou's lemma provides the converse inequality.

- This proof implies also that $S_\Omega(\beta, k)$ is continuous with respect to (β, k) in the range $0 \le \beta < kn/q$ and that the corresponding minimizer depends continuously on (β, k) in $L^q(\Omega)$ and $H_0^1(\Omega)$.

3 Semi-linear case ($\beta > kn/q$) : non-compact minimizing sequence

When $\beta > kn/q$, the problem (1) is under the total influence of the linear problem (8). Let us recall that its minimizer S is independent of the smooth bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ and that this minimizing problem has no solution. According to [2], a minimizing sequence of (8) is given by $\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon$ where :

$$u_\varepsilon(x) = \frac{\varepsilon^{\frac{n-2}{4}}\zeta(x)}{(\varepsilon + |x|^2)^{\frac{n-2}{2}}} \tag{11}$$

with $\zeta \in C^\infty(\bar{\Omega}; [0,1])$ is a smooth compactly supported cutoff function that satisfy $\zeta(x) = 1$ in a small neighborhood of the origin in Ω. Recall that $\frac{n-2}{2} = n/q$. Recall that $(k+1)(n-2) > kn/q$ for any $k \geq 0$. We know from [2] that

$$\|\nabla u_\varepsilon\|_{L^2}^2 = K_1 + O(\varepsilon^{\frac{n-2}{2}}), \qquad \|u_\varepsilon\|_{L^q}^2 = K_2 + o(\varepsilon^{\frac{n-2}{2}})$$

and that $S = K_1/K_2$.

The goal of this section is the proof of the following Proposition.

Proposition 4 *If $\frac{\beta}{k} > \frac{n}{q}$, one has*

$$S_\Omega(\beta, k) = S \tag{12}$$

and the problem (1) *admits no minimizer in $H_0^1(\Omega)$. Moreover, the sequence $\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon$ defined by* (11) *is a minimizing sequence for both* (1) *and the linear problem* (8).

Proof. Suppose by contradiction that (1) is achieved by $u \in H_0^1(\Omega)$. Then $u \neq 0$ and therefore the following strict inequality holds :

$$S \leq \int_\Omega |\nabla u|^2 < I_{\Omega;\beta,k}(u) = S_\Omega(\beta, k).$$

Therefore the identity (12) implies that (1) has no minimizer. To prove (12) and the rest of the statement, it is sufficient to show that

$$I_{\Omega;\beta,k}\left(\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon\right) = S + o(1) \tag{13}$$

in the limit $\varepsilon \to 0$, because one obviously has $S \leq S_\Omega(\beta, k) \leq I_{\Omega;\beta,k}(\|u_\varepsilon\|_{L^q}^{-1} u_\varepsilon)$. The limit (13) will follow immediately from the next result. ∎

Proposition 5 *With the previous notations,* (13) *holds and more precisely, as $\varepsilon \to 0$, one has :*

$$\int_\Omega |x|^\beta |u_\varepsilon|^k |\nabla u_\varepsilon|^2 dx = \begin{cases} C\varepsilon^{\frac{2\beta - k(n-2)}{4}} + o\left(\varepsilon^{\frac{2\beta-k(n-2)}{4}}\right) & \text{if } \frac{kn}{q} < \beta < (k+1)(n-2) \\ O\left(\varepsilon^{\frac{(k+2)(n-2)}{4}} |\log\varepsilon|\right) & \text{if } \beta = (k+1)(n-2) \\ O\left(\varepsilon^{\frac{(k+2)(n-2)}{4}}\right) & \text{if } \beta > (k+1)(n-2) \end{cases} \tag{14}$$

with $C = \displaystyle\int_{\mathbb{R}^n} \frac{|x|^{\beta+2}}{(1+|x|^2)^{\frac{kn-2}{2}+n}} dx$ *and thus :*

$$I_{\Omega;\beta,k}\left(\frac{u_\varepsilon}{\|u_\varepsilon\|_{L^q}}\right) = S + \begin{cases} C\varepsilon^{\frac{2\beta-k(n-2)}{4}} K_2 + o(\varepsilon^{\frac{2\beta-k(n-2)}{4}}) & \text{if } \frac{kn}{q} < \beta < (k+1)(n-2) \\ O(\varepsilon^{\frac{(k+2)(n-2)}{4}} |\log\varepsilon|) & \text{if } \beta = (k+1)(n-2) \\ O(\varepsilon^{\frac{(k+2)(n-2)}{4}}) & \text{if } \beta > (k+1)(n-2). \end{cases} \tag{15}$$

Proof. The only verification is that of (14).

$$\begin{aligned}\int_\Omega |x|^\beta |u_\varepsilon|^k |\nabla u_\varepsilon|^2 dx &= (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_\Omega \frac{|\zeta(x)|^{k+2}|x|^{\beta+2}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}} dx \\ &+ \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_\Omega \frac{|\zeta(x)|^k |\nabla\zeta(x)|^2 |x|^\beta}{(\varepsilon+|x|^2)^{\frac{(k+2)(n-2)}{2}}} dx \\ &- 2(n-2)\varepsilon^{\frac{(k+2)(n-2)}{4}} \int_\Omega \frac{|\zeta(x)|^{k+1}|x|^\beta \nabla\zeta(x).x}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n-1}} dx.\end{aligned}$$

Since $\zeta \equiv 1$ on a neighborhood of a and using the Dominated Convergence Theorem, a direct computation gives

$$\begin{aligned}\int_\Omega |x|^\beta |u_\varepsilon|^k |\nabla u_\varepsilon|^2 dx &= (n-2)^2 \varepsilon^{\frac{(k+2)(n-2)}{4}} \int_\Omega \frac{|\zeta(x)|^{k+2}|x|^{\beta+2}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}} dx \\ &+ O(\varepsilon^{\frac{(k+2)(n-2)}{4}}).\end{aligned}$$

Here we will consider the following three subcases.

1. Case $\beta < (k+1)(n-2)$

$$\begin{aligned}\varepsilon^{\frac{(k+2)(n-2)}{4}} \int_\Omega \frac{|\zeta(x)|^{k+2}|x|^{\beta+2}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}} dx &= \int_{\mathbb{R}^n} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}}|x|^{\beta+2}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}} dx + \int_{\mathbb{R}^n\setminus\Omega} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}}|x|^{\beta+2}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}} dx \\ &+ \int_\Omega \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}}(|\zeta(x)|^{k+2}-1)|x|^{\beta+2}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}} dx.\end{aligned}$$

Using the Dominated Convergence Theorem, and the fact that $\zeta \equiv 1$ on a neighborhood of 0, one obtains

$$\varepsilon^{\frac{(k+2)(n-2)}{4}} \int_\Omega \frac{|\zeta(x)|^{k+2}|x|^{\beta+2}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}} dx = \int_{\mathbb{R}^n} \frac{\varepsilon^{\frac{(k+2)(n-2)}{4}}|x|^{\beta+2}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}} dx + O(\varepsilon^{\frac{(k+2)(n-2)}{4}}).$$

By a simple change of variable, one gets

$$\varepsilon^{\frac{(k+2)(n-2)}{4}} \int_\Omega \frac{|\zeta(x)|^{k+2}|x|^{\beta+2}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}} dx = \varepsilon^{\frac{2\beta-k(n-2)}{4}} \int_{\mathbb{R}^n} \frac{|y|^{\beta+2}}{(1+|y|^2)^{\frac{k(n-2)}{2}+n}} dy + o(\varepsilon^{\frac{2\beta-k(n-2)}{4}})$$

which gives (14) in this case.

2. Case $\beta=(k+1)(n-2)$

$$\begin{aligned}\int_\Omega |x|^\beta |u_\varepsilon|^k|\nabla u_\varepsilon|^2dx &= (n-2)^2\varepsilon^{\frac{(k+2)(n-2)}{4}}\int_\Omega \frac{|\zeta(x)|^{k+2}|x|^{k(n-2)+n}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}}dx+O(\varepsilon^{\frac{(k+2)(n-2)}{4}})\\ &= (n-2)^2\varepsilon^{\frac{(k+2)(n-2)}{4}}\int_\Omega \frac{(|\zeta(x)|^{k+2}-1)|x|^{k(n-2)+n}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}}dx\\ &\quad +(n-2)^2\varepsilon^{\frac{(k+2)(n-2)}{4}}\int_\Omega \frac{|x|^{k(n-2)+n}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}}dx+O(\varepsilon^{\frac{(k+2)(n-2)}{4}})\\ &= (n-2)^2\varepsilon^{\frac{(k+2)(n-2)}{4}}\int_\Omega \frac{|x|^{k(n-2)+n}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}}dx+O(\varepsilon^{\frac{(k+2)(n-2)}{4}})\end{aligned}$$

One has, for some constants $R_1<R_2$:

$$\int_{B(0,R_1)}\frac{|x|^{k(n-2)+n}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}}dx\le\int_\Omega \frac{|x|^{k(n-2)+n}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}}dx\le\int_{B(0,R_2)}\frac{|x|^{k(n-2)+n}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}}dx$$

with

$$\begin{aligned}\int_{B(0,R)}\frac{|x|^{k(n-2)+n}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}}dx &= \omega_n\int_0^R \frac{r^{k(n-2)+2n-1}}{(\varepsilon+r^2)^{k\frac{(n-2)}{2}+n}}dr\\ &= \frac{1}{2}\omega_n|\log\varepsilon|+O(1).\end{aligned}$$

Consequently, one has :

$$\int_\Omega |x|^\beta|u_\varepsilon|^k|\nabla u_\varepsilon|^2dx = O\left(\varepsilon^{\frac{(k+2)(n-2)}{4}}|\log\varepsilon|\right).$$

3. Case $\beta>(k+1)(n-2)$

$$\int_\Omega |x|^\beta|u_\varepsilon|^k|\nabla u_\varepsilon|^2dx=(n-2)^2\varepsilon^{\frac{(k+2)(n-2)}{4}}\int_\Omega \frac{|\zeta(x)|^{k+2}|x|^{\beta+2}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}}dx+O(\varepsilon^{\frac{(k+2)(n-2)}{4}}).$$

One can apply the Dominated Convergence Theorem :

$$\frac{|\zeta(x)|^{k+2}|x|^{\beta+2}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}}\longrightarrow|\zeta(x)|^{k+2}|x|^{\beta-(k(n-2)+2n-2)}\quad\text{when}\quad\varepsilon\to0$$

and

$$\frac{|\zeta(x)|^{k+2}|x|^{\beta+2}}{(\varepsilon+|x|^2)^{\frac{k(n-2)}{2}+n}}\le|\zeta(x)|^{k+2}|x|^{\beta-(k(n-2)+2n-2)}\in L^1(\Omega).$$

So, it follows that

$$\int_\Omega |x|^\beta|u_\varepsilon|^k|\nabla u_\varepsilon|^2dx=O(\varepsilon^{\frac{(k+2)(n-2)}{4}})$$

which again is (14). ∎

4 The critical case ($\beta = kn/q$) : non-existence of smooth minimizers

The critical case is a natural generalization of the well known problem with $\beta = k = 0$. In this section, the following result will be established.

Proposition 6 *If $\beta = kn/q$, one has*

$$S_\Omega(\beta, k) = S_{\widetilde{\Omega}}(\beta, k) \tag{16}$$

for any two smooth neighborhoods $\Omega, \widetilde{\Omega} \subset \mathbb{R}^n$ of the origin. Moreover, if Ω is star-shaped around $x = 0$, the minimization problem (1) *admits no solution in the class :*

$$H_0^1 \cap H^{3/2} \cap L^\infty(\Omega).$$

If $k < 1$, the negative result holds, provided additionally $u^{k-1} \in L^n(\Omega)$.

The rest of this section is devoted to the proof of this statement. Note that if the minimization problem (1) had a minimizer u with non constant sign in this class of regularity, then $|u|$ would be a positive minimizer in the same class, thus it is sufficient to show that there are no positive minimizers.

4.1 $S_\Omega(\beta, k)$ does not depend on the domain

If $\Omega \subset \Omega'$, there is a natural injection $i : H_0^1(\Omega) \hookrightarrow H_0^1(\Omega')$ that corresponds to the process of extension by zero. Let $u_j \in H_0^1(\Omega)$ be a minimizing sequence for $S_\Omega(\beta, k)$. Then $\|i(u_j)\|_{L^q(\Omega')} = 1$ thus

$$S_{\Omega'}(\beta, k) \leq I_{\Omega';\beta,k}(i(u_j)) = I_{\Omega;\beta,k}(u_j)$$

and therefore $S_{\Omega'}(\beta, k) \leq S_\Omega(\beta, k)$.

Conversely, let us now consider the scaling transformation (6) which, in the case of $\frac{\beta}{k} = \frac{n}{q}$, leaves both $\|u\|_{L^q(\Omega)}$ and $I_{\Omega;\beta,k}(u)$ invariant. If u_j is a minimizing sequence on Ω then $v_j = u_{j,\lambda^{-1}}$ is an admissible sequence on Ω_λ thus :

$$S_{\Omega_\lambda}(\beta, k) \leq I_{\Omega_\lambda;\beta,k}(v_j) = I_{\Omega;\beta,k}(u_j) \to S_\Omega(\beta, k).$$

Conversely, if v_j is a minimizing sequence on Ω_λ then $u_j = v_{j,\lambda}$ is an admissible sequence on Ω and :

$$S_\Omega(\beta, k) \leq I_{\Omega;\beta,k}(u_j) = I_{\Omega_\lambda;\beta,k}(v_j) \to S_{\Omega_\lambda}(\beta, k).$$

This ensures that $S_{\Omega_\lambda}(\beta, k) = S_\Omega(\beta, k)$ for any $\lambda > 0$.

Finally, given two smooth bounded open subsets Ω and $\widetilde{\Omega}$ of $\mathbb{R}^n$ that both contain 0, one can find $\lambda, \mu > 0$ such that $\Omega_\lambda \subset \widetilde{\Omega} \subset \Omega_\mu$ and the previous inequalities read

$$S_{\Omega_\mu}(\beta, k) \leq S_{\widetilde{\Omega}}(\beta, k) \leq S_{\Omega_\lambda}(\beta, k) \qquad \text{and} \qquad S_\Omega(\beta, k) = S_{\Omega_\lambda}(\beta, k) = S_{\Omega_\mu}(\beta, k)$$

thus ensuring $S_\Omega(\beta, k) = S_{\widetilde{\Omega}}(\beta, k)$.

4.2 Pohozaev identity and the non-existence of smooth minimizers

Suppose by contradiction that a bounded minimizer u of (1) exists for some star-shaped domain Ω with $\beta = kn/q$, i.e. $u \in H_0^1 \cap L^\infty(\Omega)$. As mentioned in the introduction $|u|$ is also a minimizer thus, without loss of generality, one can also assume that $u \geq 0$. Moreover, u will satisfy the Euler-Lagrange equation (3) in the weak sense, for any test-function in $H_0^1 \cap L^\infty(\Omega)$.

In the following argument, inspired by [13], one will use $(x\cdot\nabla)u$ and u as test functions. The later is fine but the former must be checked out carefully. A brutal assumption like $(x\cdot\nabla)u \in H_0^1\cap L^\infty(\Omega)$ is much too restrictive. Let us assume instead that

$$u \in H_0^1 \cap H^{3/2} \cap L^\infty \qquad \text{and (if } k<1\text{) } \; u^{k-1} \in L^n(\Omega). \tag{17}$$

Note that if $v \in H^{3/2}$ then $|v| \in H^{3/2}$ thus the assumption $u \geq 0$ still holds without loss of generality. Then one can find a sequence $\phi_n \in H_0^1 \cap L^\infty(\Omega)$ such that $\phi_n \to \phi = (x\cdot\nabla)u$ in $H^{1/2}(\Omega)$ and almost everywhere and such that each sequence of integrals converges to the expected limit :

$$(-\Delta u|\phi_n) \to (-\Delta u|\phi), \qquad (u^k|\phi_n) \to (u^k|\phi)$$
$$(u^{k-1}\nabla u|\phi_n) \to (u^{k-1}\nabla u|\phi) \qquad \text{and} \qquad (u^{q-1}|\phi_n) \to (u^{q-1}|\phi).$$

Indeed, each integral satisfies a domination assumption :

$$|(-\Delta u|\phi_n-\phi)| \leq \|u\|_{H^{3/2}} \|\phi_n-\phi\|_{H^{1/2}},$$
$$|(u^k|\phi_n-\phi)| \leq \|u^k\|_{L^{2n/(n+1)}} \|\phi_n-\phi\|_{L^{2n/(n-1)}} \leq C_\Omega \|u\|_{L^\infty}^k \|\phi_n-\phi\|_{H^{1/2}},$$
$$|(u^{k-1}\nabla u|\phi_n-\phi)| \leq \begin{cases} \|u\|_{L^\infty}^{k-1} \|\nabla u\|_{L^2} \|\phi_n-\phi\|_{L^2} & \text{if } k\geq 1,\\ \|u^{k-1}\|_{L^n} \|\nabla u\|_{L^{2n/(n-1)}} \|\phi_n-\phi\|_{L^{2n/(n-1)}} & \\ \qquad \leq C_\Omega \|u^{k-1}\|_{L^n} \|u\|_{H^{3/2}} \|\phi_n-\phi\|_{H^{1/2}} & \text{if } k<1,\end{cases}$$
$$|(u^{q-1}|\phi_n-\phi)| \leq \|u^{q-1}\|_{L^{2n/(n+1)}} \|\phi_n-\phi\|_{L^{2n/(n-1)}} \leq C_\Omega \|u\|_{L^\infty}^{q-1} \|\phi_n-\phi\|_{H^{1/2}}.$$

Thus, the Euler-Lagrange is also satisfied in the weak sense for the test-function $\phi=(x\cdot\nabla)u$.

Let us multiply by $(x\cdot\nabla)u$ and integrate by parts :

$$-\int_\Omega \operatorname{div}\left(p(x,u)\nabla u\right)\times(x\cdot\nabla)u + \frac{k}{2}\int_\Omega |x|^\beta |u|^{k-2}|\nabla u|^2 u(x\cdot\nabla)u = S(\beta,k)\int_\Omega |u|^{q-2}u(x\cdot\nabla)u.$$

An integration by part in the right-hand side and the condition $u\in H_0^1(\Omega)$ provide :

$$S(\beta,k)\int_\Omega |u|^{q-2}u(x\cdot\nabla)u = -S(\beta,k)\,\frac{n-2}{2}\int_\Omega |u|^q = -\frac{n}{q}S(\beta,k).$$

The first term of the left-hand side is :

$$-\int_\Omega \operatorname{div}\left(p(x,u)\nabla u\right)\times(x\cdot\nabla)u = B(u) + \int_\Omega p(x,u)|\nabla u|^2 - \int_{\partial\Omega} p(x,u)\,(x\cdot\nabla)u\,\frac{\partial u}{\partial\nu}$$

with $B(u)$ define as follows and dealt with by a second integration by part

$$\begin{aligned} B(u) &= \sum_{i,j}\int_\Omega x_j\left(1+|x|^\beta|u|^k\right)(\partial_i u)(\partial_i\partial_j u)\\ &= -B(u) - n\int_\Omega p(x,u)|\nabla u|^2 - \beta\int_\Omega |x|^\beta|u|^k|\nabla u|^2\\ &\quad - k\int_\Omega |x|^\beta|u|^{k-2}|\nabla u|^2u(x\cdot\nabla)u + \int_{\partial\Omega} p(x,u)|\nabla u|^2(x\cdot\mathbf{n}). \end{aligned}$$

On the boundary, $p(x,u)=1$ and as $u\in H_0^1(\Omega)$, one has also $\nabla u = \frac{\partial u}{\partial\nu}\mathbf{n}$ where $\mathbf{n}$ denotes the normal unit vector to $\partial\Omega$ and in particular $|\nabla u| = |\frac{\partial u}{\partial\nu}|$, thus

$$B(u) = -\frac{n}{2}\int_\Omega p(x,u)|\nabla u|^2 - \frac{\beta}{2}\int_\Omega |x|^\beta|u|^k|\nabla u|^2 - \frac{k}{2}\int_\Omega |x|^\beta|u|^{k-2}|\nabla u|^2u(x\cdot\nabla)u + \frac{1}{2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2(x\cdot\mathbf{n}).$$

The whole energy estimate with $(x \cdot \nabla)u$ boils down to :

$$\frac{n-2}{2}\int_\Omega p(x,u)|\nabla u|^2 + \frac{\beta}{2}\int_\Omega |x|^\beta |u|^k |\nabla u|^2 + \frac{1}{2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right|^2 (x\cdot \mathbf{n}) = \frac{n}{q}S(\beta,k).$$

Finally, to deal with the first term, let us multiply (3) by u and integrate by parts ; one gets :

$$\int_\Omega p(x,u)|\nabla u|^2 = \int_\Omega (1+|x|^\beta|u|^k)|\nabla u|^2 = -\frac{k}{2}\int_\Omega |x|^\beta|u|^k|\nabla u|^2 + S(\beta,k).$$

Combining both estimates provides :

$$\frac{1}{2}\left(\beta - \frac{kn}{q}\right)\int_\Omega |x|^\beta|u|^k|\nabla u|^2 + \frac{1}{2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial\nu}\right|^2 (x\cdot\mathbf{n}) = 0. \tag{18}$$

As $\beta = kn/q$ and $x \cdot \mathbf{n} > 0$ (Ω is star-shaped), one gets $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$.

The Euler-Lagrange equation (3) now reads :

$$-p(x,u)\Delta u = \frac{k}{2}|x|^\beta|u|^{k-2}u|\nabla u|^2 + \beta|x|^{\beta-2}|u|^k(x\cdot\nabla)u + \mu|u|^{q-2}u$$

which for $u \geq 0$ boils down to

$$\begin{aligned}-p(x,u)\Delta u &= |x|^{\beta-2}u^{k-1}\left(\frac{k}{2}|x|^2|\nabla u|^2 + u(x\cdot\nabla)u\right) + \mu u^{q-1}\\ &= |x|^{\beta-2}u^{k-1}\left(\sqrt{\frac{k}{2}}|x|\nabla u + Cux\right)^2 - C^2|x|^\beta u^{k+1} + \mu u^{q-1}\end{aligned}$$

with $2\sqrt{k/2}C = \beta$. For any $t \in \mathbb{R}$, one has therefore :

$$-\Delta u + tu = \frac{|x|^{\beta-2}u^{k-1}}{p(x,u)}\left(\sqrt{\frac{k}{2}}|x|\nabla u + Cux\right)^2 + \frac{\mu u^{q-1}}{p(x,u)} + tu - \frac{C^2|x|^\beta u^{k+1}}{p(x,u)} = f(t,x).$$

As $u \in L^\infty$, one can chose $t > C^2|x|^\beta\,\|u\|_{L^\infty}^k$. Then $f(t,x) \geq 0$ and the maximum principle implies that either $u = 0$ or $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$. In particular, only the solution $u = 0$ satisfies simultaneously Dirichlet and Neumann boundary conditions, which leads to a contradiction because $\|u\|_{L^q} = 1$. ∎

Remarks

1. Note that Pohozaev identity (18) prevents the existence of minimizers when $\beta \geq kn/q$. However, the technique we used in §3 (when $\beta > kn/q$) enlightens the leading term of the problem and avoids dealing with artificial regularity assumptions.

2. Similarly, one could check that the computation is also correct if

$$u \in H_0^1 \cap H^2 \cap L^\infty(\Omega) \qquad \text{and (if } k<1\text{)}\ \ u^{k-1} \in L^{n/2}. \tag{19}$$

   Assumption (19) is only preferable over (17) for $k < 1$. But it requires additional regularity in the interior of Ω and would not allow to assume $u \geq 0$ without loss of generality because in general, $v \in H^2 \not\Rightarrow |v| \in H^2$.

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